

# REDUCTIVE AND UNIPOTENT ACTIONS OF AFFINE GROUPS

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**ABSTRACT.** We present a generalized version of classical geometric invariant theory *à la Mumford* where we consider an affine algebraic group  $G$  acting on a specific affine algebraic variety  $X$ . We define the notions of *linearly reductive* and of *unipotent action* in terms of the  $G$  fixed point functor in the category of  $(G, \mathbb{k}[X])$ -modules. In the case that  $X = \{\star\}$  we recuperate the concept of linearly reductive and of unipotent group. We prove in our “relative” context some of the classical results of GIT such as: existence of quotients, finite generation of invariants, Kostant–Rosenlicht’s theorem and Matsushima’s criterion. We also present a partial description of the geometry of such *linearly reductive* actions.

## 1. INTRODUCTION

Given a fixed affine algebraic group  $G$ , classical geometric invariant theory *à la Mumford*, is the study of the interplay between the following three different mathematical themes: the geometric properties of the actions of the group  $G$  on algebraic varieties, the structure of the category  ${}_G\mathcal{M}$  of all rational  $G$ -modules and thirdly the inner structure of the group  $G$  – or of its Lie algebra.

Even though this theory has been extremely successful in the degree of generality that was originally conceived for, it has the handicap of not being very sensitive to the particular properties of a specific action.

In this paper we develop the initial features of what we call a *relative theory*. We will be concerned with pairs  $(G, X)$  where  $X$  is a variety equipped with a  $G$ -action, and the group  $G$  acts “reductively” in the category of  $(G, \mathbb{k}[X])$ -modules, even though it need not be *reductive*. In this context (weaker than the standard one) we still obtain for  $X$  and other varieties naturally related to it, some of the usual results of classical invariant theory.

The following theorem, due to E. Cline, B. Parshall and L. Scott and that appeared in [2], conspicuously illuminates this perspective of the theory and was a strong motivation for the considerations that follow.

*Let  $G$  be an affine algebraic group and  $K \subseteq G$  a closed subgroup, then the homogeneous space  $G/K$  is an affine variety if and only if all the exact sequences in the category  $({}_K, {}_G)\mathcal{M}$  split in  ${}_K\mathcal{M}$  — in the language of the present paper, if and only if the action of  $K$  on  $G$  is linearly reductive.*

In parallel with the development of the concept of linearly reductive action of  $G$  on  $X$  it is natural to define in the same context, the notion of unipotent action. For unipotent actions we generalize the classical Kostant–Rosenlicht result on the closedness of the orbits of  $G$  in  $X$ .

If we apply the definitions of linearly reductive action or of unipotent action (see Definitions 2.1 and 7.1) to the situation of a pair  $(G, \{\star\})$  where  $\{\star\}$  is the one point variety, we obtain the concept of linearly reductive or of unipotent group.

As we are dealing with the properties of *pairs*  $(G, X)$ , one of the main advantages of the relative point of view is that we can formulate and prove *transitivity* results as the following:

*Let  $H \subseteq G$  be a closed subgroup of the group  $G$  and assume that the action of  $G$  on  $X$  is linearly reductive, and that the action of  $H$  on  $G$  is also linearly reductive, then the action of  $H$  on  $X$  is linearly reductive.*

This result, when applied to the case that  $X$  is the variety consisting of a single point, yields the Matsushima’s criterion in characteristic zero, see for example [16, 2], and in such a way the classical Matsushima’s criterion can be profitably interpreted as a transitivity result.<sup>1</sup>

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*Date:* June 18, 2014.

The authors would like to thank Csic-UDELAR and Anii for their partial support.

<sup>1</sup>We plan to develop in future work, the theory of *geometrically reductive actions*, this will guarantee the above result for the situation of positive characteristic.

Next, we present a brief description of the contents of this paper.

In Section 2 we establish in Definitions 2.1 the concepts of linearly reductive action of an affine algebraic group  $G$  on a commutative and rational  $G$ -module algebra  $R$ , and in parallel on an affine algebraic variety  $X$ .

In Theorem 2.4 we present different characterizations of the concept of a linearly reductive action, some of them with a cohomological flavor and others in terms of total integrals –compare with [4, 5, 6]– and of relative splittings. A version of the concept of Reynolds operator also appears related to the linear reductivity of the action.

In the particular case of a linearly reductive action of a unipotent group  $U$ , we show that the target variety  $X$  is  $U$ -isomorphic with  $U \times S$  where  $S$  is affine and acted trivially by the group. Hence, up to trivial factors, the only linearly reductive action of a unipotent group is the action of the group on itself by translations, see Lemma 2.10 and Theorem 2.11.

In Section 3 we consider the case of a group  $G$  and a closed subgroup  $K \subseteq G$  acting by translations. We encounter the well known theorem of Cline, Parshall and Scott that in our language means that the action of  $K$  on  $G$  is linearly reductive if and only if  $G/K$  is an affine variety.

We finish the section proving that in the same situation than above, the linear reductivity of the action of  $K$  on  $G$  together with the linear reductivity of the action of  $G$  on  $G/K$  are equivalent to the linear reductivity of  $K$  (Theorem 3.5).

In Section 4 we deal with transitivity results. In Theorem 4.1, we prove the generalization of Matsushima's criterion considered above. Moreover in the same theorem we prove that in the case that  $K \triangleright G$  is a normal subgroup and  $R$  is a  $G$ -module algebra, the linear reductivity of the action of  $K$  on  $R$  and that of the action of  $G/K$  on  ${}^K R$ , are equivalent to that of the action of  $G$  on  $R$ .

In Section 5 we use the above mentioned transitivity results in order to prove a result similar to – but weaker than – the classical Levi decomposition of an algebraic group, and use it to give a characterization of linearly reductive actions of a group  $G$  on a variety  $X$  (see Theorem 5.5).

In Section 6 we prove that the two main initial results of classical geometric invariant theory hold for our weaker context of linearly reductive *actions*: the finite generation of the invariants, and the existence of semi-geometric quotients, see Theorems 6.1 and 6.3.

In Section 7, we define the relative concept of unipotent action of an affine algebraic group on an affine variety  $X$ . This is done in terms of the faithfulness of the  $G$ -fixed point functor in the category  $(G, \mathbb{k}[X])\mathcal{M}$ . We prove that an action that is unipotent and linearly reductive reduces to the action of  $G$  on itself by translations, and we also establish a generalization of Kostant–Rosenlicht theorem: if the action of  $G$  on  $X$  is unipotent, then the orbits of the action of  $G$  on  $X$  are closed (see Theorem 7.5).

## NOTATIONS:

All the geometric and algebraic objects will be defined over an algebraically closed field  $\mathbb{k}$  and all the algebras considered will be commutative. The geometric actions are on the right and the algebraic actions on the left.

If  $R \in {}_G\mathcal{M}$  is a rational commutative  $G$ -module algebra, then  $(G, R)\mathcal{M}$  is the category of rational  $(G, R)$ -modules whose objects  $M$  are rational  $G$ -modules and  $R$ -modules subject to the following compatibility condition: if  $x \in G, r \in R, m \in M$ , then  $x \cdot (rm) = (x \cdot r)(x \cdot m)$ , with the obvious morphisms.

Given a right regular action  $X \times G \rightarrow X$  of  $G$  on the affine variety  $X$  (i.e.  $X$  is a *regular  $G$ -variety* or simply a  *$G$ -variety*), then  $R = \mathbb{k}[X]$  is a rational  $G$ -module algebra and we write  $(K, X)\mathcal{M}$  instead of  $(G, \mathbb{k}[X])\mathcal{M}$ .

If  $M$  is a rational  $G$ -module, then  $M \mapsto {}^G M \subseteq M$  is the covariant left exact  $G$ -fixed part functor  ${}_G\mathcal{M} \rightarrow {}_G\mathcal{M}$ .

Rational  $G$ -modules are  $\mathbb{k}[G]$ -comodules with (right) coaction  $\chi : M \rightarrow M \otimes \mathbb{k}[G]$  characterized by the condition:  $\chi(m) = \sum m_0 \otimes m_1 \in M \otimes \mathbb{k}[G]$ , if and only if  $x \cdot m = \sum m_0 m_1(x)$  for all  $x \in G$  – we use Sweedler's notation. If  $G, R$  are as above, then  $M \in (G, R)\mathcal{M}$ , if and only if  $\chi_M(r \cdot m) = \sum r_0 \cdot m_0 \otimes r_1 m_1$ .

If  $G$  is an affine algebraic group  $\mathcal{R}_u(G)$  denotes the unipotent radical of  $G$ . In case that the unipotent radical is trivial we say the  $G$  is reductive and it is well known that this is equivalent to be linearly reductive in characteristic zero.

## 2. REDUCTIVE ACTIONS

The following definition of *linearly reductive action* will be central to our considerations.

### Definition 2.1.

- (1) Let  $G$  be an affine algebraic group and  $R$  a rational  $G$ -module algebra. We say that the action of  $G$  on  $R$  is *linearly reductive* if for every triple  $(M, J, \lambda)$  where  $M \in {}_{(G,R)}\mathcal{M}$ ,  $J \subseteq R$  is a  $G$ -stable ideal and  $\lambda : M \rightarrow R/J$  is a surjective morphism of  $(G, R)$ -modules; there exists an element  $m \in {}^G M$ , such that  $\lambda(m) = 1 + J \in R/J$ . In the context above, if the action of  $G$  on  $R$  is given, we say that  $(G, R)$  is a linearly reductive pair.
- (2) In the case that  $R = \mathbb{k}[X]$  and the action of  $G$  on  $R$  is linearly reductive we say that the action of  $G$  on  $X$  is linearly reductive and also that the pair  $(G, X)$  is linearly reductive.

**Definition 2.2.** Let  $G$  be an affine algebraic group and  $R$  a rational  $G$ -module algebra. A *(left) integral* for  $G$  with values in  $R$  is an  $G$ -equivariant linear map  $\sigma : \mathbb{k}[G] \rightarrow R$ . The integral  $\sigma$  is said to be *total* if  $\sigma(1) = 1$ .

### Observation 2.3.

- (1) The  $G$ -equivariance condition  $\sigma(x.f) = x.\sigma(f)$  of the integral  $\sigma$ , can be expressed in terms of the commutativity of the diagram below:

$$\begin{array}{ccc} \mathbb{k}[G] & \xrightarrow{\sigma} & R \\ \Delta \downarrow & & \downarrow \chi \\ \mathbb{k}[G] \otimes \mathbb{k}[G] & \xrightarrow{\sigma \otimes \text{id}} & R \otimes \mathbb{k}[G] \end{array} \quad \text{i.e.} \quad \sum \sigma(f_1) \otimes f_2 = \sum \sigma(f)_0 \otimes \sigma(f)_1,$$

where  $\Delta$  is the comultiplication in  $\mathbb{k}[G]$  and  $\chi$  is the coaction in  $R$ .

- (2) Recall that in the situation above there exists a total integral for  $G$  with values in  $R$ , if and only if  $R$  is injective as an object in  ${}_G\mathcal{M}$  (see for example [2, 4, 5, 6, 8]).

**Theorem 2.4.** Let  $G$  an affine algebraic group and  $R$  a rational  $G$ -module algebra. Then, the following conditions are equivalent:

- (1) The action of  $G$  on  $R$  is linearly reductive.
- (2) If  $\varphi : M \rightarrow N$  is a surjective morphism in the category  ${}_{(G,R)}\mathcal{M}$ , then  $\varphi({}^G M) = {}^G N$ .
- (3) If  $M \in {}_{(G,R)}\mathcal{M}$ , then the rational cohomology groups  $H^n(G, M) = \{0\}$  for all  $n \neq 0$  (the functors  $M \mapsto H^n(G, M)$  are the derived functors of  $M \mapsto {}^G M$ , see [11, 15]).
- (4) There exists a total integral  $\sigma : \mathbb{k}[G] \rightarrow R$ .
- (5) The  $G$ -module algebra  $R$  is an injective object in the category  ${}_G\mathcal{M}$ .
- (6) Every object  $M \in {}_{(G,R)}\mathcal{M}$  is injective in  ${}_G\mathcal{M}$ .

*Proof.* First we prove that the first two conditions are equivalent. It is clear that the second condition implies the first. Conversely, let  $M, N, \varphi$  and  $n \in {}^G N$  be as in (2), consider  $m \in M$  such that  $\varphi(m) = n$  and call  $\overline{M} = \{\sum_i t_i(x_i \cdot m) : t_i \in R, x_i \in G\}$  the  $(G, R)$ -submodule of  $M$  generated by  $m$ . Clearly the ideal  $J = \{r \in R : rn = 0\} \subseteq R$  is  $G$ -stable and we can define the map  $\lambda : \overline{M} \rightarrow R/J$  by the rule  $\lambda(\overline{m}) = \sum_i t_i + J$ . Notice that this definition makes sense: if we have other representation  $\overline{m} = \sum_j u_j(y_j \cdot m)$ , then applying  $\varphi$  we obtain that  $(\sum_i t_i)n = (\sum_j u_j)n$  and then  $\sum_i t_i - \sum_j u_j \in J$ . It is also clear that  $\lambda$  is a morphism in the category  ${}_{(G,R)}\mathcal{M}$ . Using our hypothesis we find  $\overline{m}_0 \in {}^G \overline{M}$ , such that  $\lambda(\overline{m}_0) = 1 + J$ . Then, writing  $\overline{m}_0 = \sum_j t_j(x_j \cdot m)$ , we have that  $\sum_i t_i \equiv 1 \pmod{J}$  i.e.  $\varphi(\overline{m}_0) = (\sum_i t_i)n = n$ . The equivalence of (2) and (3) is clear. The equivalence of (4) and (5) was already mentioned in part (2) of Observation 2.3, and

it is obvious that (6) implies (5). The proof that (4) implies (2) goes as follows. For an arbitrary object  $M \in {}_{(G,R)}\mathcal{M}$  we can consider the map  $p_M : M \rightarrow M$  defined as:

$$p_M(m) = \sum \sigma(\mathcal{S}m_1) \cdot m_0,$$

where  $\mathcal{S} : \mathbb{k}[G] \rightarrow \mathbb{k}[G]$  is the antipode map  $\mathcal{S}(f)(k) = f(k^{-1})$ ,  $k \in G$ . By direct computation one can prove that : if  $m \in {}^G M$ , then  $p_M(m) = m$ ;  $p_M(M) = {}^G M$  and for an arbitrary morphism  $\varphi : M \rightarrow N$  in  ${}_{(G,R)}\mathcal{M}$  the diagram below commutes.

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ p_M \downarrow & & \downarrow p_N \\ {}^G M & \xrightarrow{\varphi|_{{}^G M}} & {}^G N. \end{array}$$

Then it is clear that if  $\varphi$  is surjective, so is  $\varphi|_{{}^G M}$ .

We finish by proving that (1) implies (6). Assume that  $V \subseteq W$  is an inclusion in the category  ${}_G\mathcal{M}$ ,  $M \in {}_{(G,R)}\mathcal{M}$  and let  $\varphi : V \rightarrow M$  be a morphism of  $G$ -modules. We want to show that it can be extended to a morphism of  $G$ -modules,  $W \rightarrow M$ . Assume that  $W$  is finite dimensional. Consider  $\text{Hom}_{\mathbb{k}}(W, M)$  and  $\text{Hom}_{\mathbb{k}}(V, M)$  as objects in  ${}_{(G,R)}\mathcal{M}$  in the following manner: if  $x \in G$ ,  $r \in R$  and  $\psi \in \text{Hom}_{\mathbb{k}}(W, M)$ , then  $(x \cdot \psi)(w) = x \cdot \psi(x^{-1} \cdot w)$  and  $(r\psi)(w) = r\psi(w)$  for  $w \in W$ , and similarly for  $\text{Hom}_{\mathbb{k}}(V, M)$ . The map  $\Gamma : \text{Hom}_{\mathbb{k}}(W, M) \rightarrow \text{Hom}_{\mathbb{k}}(V, M)$  given by the restriction of the functions from  $W$  to  $V$  is clearly a surjective morphism in  ${}_{(G,R)}\mathcal{M}$ . Then, by hypothesis, for some  $\hat{\varphi} \in {}^G \text{Hom}_{\mathbb{k}}(W, M)$  we have:  $\Gamma(\hat{\varphi}) = \varphi \in {}^G \text{Hom}_{\mathbb{k}}(V, M)$  and hence the map  $\varphi$  can be extended from  $V$  to  $W$ .

In the case that  $W$  is not necessarily finite dimensional, using standard arguments we may assume that we are in the situation that we have extended the map  $\varphi : V \rightarrow R$  maximally to  $V_{\infty}$  as shown in the diagram below.

$$\begin{array}{ccccc} V & \xrightarrow{\subseteq} & V_{\infty} & \xrightarrow{\subseteq} & W \\ \downarrow \varphi & & \swarrow \varphi_{\infty} & & \\ M & & & & \end{array}$$

If  $V_{\infty} \neq W$ , take  $w \in W \setminus V_{\infty}$  and consider  $\langle G \cdot w \rangle$ , the finite dimensional  $G$ -module generated by  $w$ . As we just proved, we can extend the restriction of  $\varphi_{\infty}$  from  $V_{\infty} \cap \langle G \cdot w \rangle$  to  $\langle G \cdot w \rangle$ . Putting together this extension of  $\varphi$  with the compatible extension  $\varphi_{\infty}$  we construct an extension of  $\varphi$  to  $V_{\infty} + \langle G \cdot w \rangle$ , and this is clearly a contradiction.  $\square$

**Observation 2.5.** It can be easily proved that the three conditions below are also equivalent the linear reductivity of the action of  $G$  on  $R$ .

- (1) If  $\theta : V \rightarrow W$  is a surjective morphism in the category  ${}_G\mathcal{M}$ , then  $(\text{id} \otimes \theta)({}^G(R \otimes V)) = {}^G(R \otimes W)$ .
- (2) If  $V \in {}_G\mathcal{M}$ , then  $H^n(G, R \otimes V) = \{0\}$  for all  $n \neq 0$ .
- (3) Every inclusion  $N \subseteq M$  in the category  ${}_{(G,R)}\mathcal{M}$  splits in  ${}_G\mathcal{M}$ .

**Observation 2.6.** (1) A natural family of morphisms  $\{p_M : M \in {}_{(G,R)}\mathcal{M}\}$  as the one constructed in the above proof is called a family of *Reynolds operators* for the action of  $G$  on  $R$ . It can be proved that the existence of a family of Reynolds operators is equivalent to the linear reductivity of the action.

- (2) Assume that  $S \in {}_{(G,R)}\mathcal{M}$  is a  $(G, R)$ -module algebra. Take  $s_0 \in {}^G S$  and consider the morphism  $\ell_{s_0} : S \rightarrow S$  defined as  $\ell_{s_0}(s) = s_0 s$  for  $s \in S$ . It is clear that  $\ell_{s_0}$  is a morphism in the category  ${}_{(G,R)}\mathcal{M}$ . Using the naturality of the family of Reynolds operators we deduce that  $p_S(s_0 s) = s_0 p_S(s)$ , equality that is valid for all  $s_0 \in {}^G S$  and  $s \in S$ . This equality can also be formulated as follows: for all  $s, t \in S$ :  $p_S(p_S(s)t) = p_S(s)p_S(t)$ . In the literature the above equality is called the *Reynolds identity*.

- (3) Assume now that  $R$  is a rational  $G$ -module algebra, and that  $K \triangleleft G$  is a closed normal subgroup of  $G$ . Assume that we have a family of Reynolds operators for the category of  $(K, R)$ -modules. Fix  $g \in G$  and consider the map  $c_g : R \rightarrow R$ ,  $c_g(r) = gr$ . If we call  $R_g$  the algebra  $R$  equipped with the action  $h \cdot_g r = ghg^{-1}r$ , then from the commutativity of the diagram:

$$\begin{array}{ccc} R & \xrightarrow{p_R} & KR \\ c_g \downarrow & & \downarrow c_g \\ R_g & \xrightarrow{p_R} & KR \end{array}$$

we deduce that  $p_R(gr) = gp_R(r)$ .

**Example 2.7.**

- (1) If  $G$  is a linearly reductive group – in the classical sense – then any rational action of  $G$  on an algebra  $R$  is linearly reductive. Conversely, in the case of a trivial action of a group  $G$  on an algebra  $R$ , the linear reductivity of the action implies the linear reductivity of the group. If we have a surjective morphism in  ${}_G\mathcal{M}$ ,  $\varphi : M \rightarrow N$ , then extending scalars we obtain the surjective morphism  $\text{id} \otimes \varphi : R \otimes M \rightarrow R \otimes N$  in  $(G, R)\mathcal{M}$ . As  $(\text{id} \otimes \varphi)({}^G(R \otimes M)) = (\text{id} \otimes \varphi)(R \otimes {}^G M) = R \otimes \varphi({}^G M) = R \otimes {}^G N$ , we conclude that  $\varphi({}^G M) = {}^G N$ .
- (2) Using a change of scalars argument, and Theorem 2.4 it follows that if  $\varphi : R \rightarrow S$  a morphism of rational  $G$ -module algebras and the action of  $G$  on  $R$  is linearly reductive, then the action of  $G$  on  $S$  is linearly reductive. In particular, if  $X$  and  $Y$  are affine  $G$ -varieties and  $f : X \rightarrow Y$  is an equivariant morphism, then if  $(G, Y)$  is linearly reductive, so is  $(G, X)$ .
- (3) Let  $S = R \otimes L$ , where  $R, L$  are  $G$ -module algebras. Then, from the linear reductivity of the action of  $G$  on  $R$  we deduce the same reductivity for the action on  $S$ . Similarly if  $X, Y$  are affine  $G$ -varieties, and the action of  $G$  on  $Y$  is linearly reductive, the same is true for  $X \times Y$ .
- (4) Let  $K \triangleleft G$  be a (normal) inclusion of affine algebraic groups. Assume that  $R$  is a rational  $G$ -module algebra where  $K$  acts trivially. Then the action of  $G$  in  $R$  is linearly reductive if and only if the action of  $G/K$  in  $R$  is linearly reductive.
- (5) The action of  $G$  in itself by translations is linearly reductive. Indeed, the fundamental theorem on Hopf modules guarantees the existence of a natural equivalence in the category of  $(G, \mathbb{k}[G])$ -modules:  $\Theta_M : M \cong \mathbb{k}[G] \otimes {}^G M$ , where  $M$  is taken with its action and on the tensor product we use the diagonal action, see for example [8, Theorem 4.3.29]. This implies that a surjective morphism in the category of  $(G, \mathbb{k}[G])$ -modules, restricts to a surjective morphism between the corresponding  $G$ -fixed parts.
- (6) Let  $X$  be an affine variety and  $G$ -variety. An *equivariant cross section* is a morphism  $\varphi : X \rightarrow G$ , with the property that for all  $x \in X$  and  $k \in G$ ,  $\varphi(x \cdot k) = \varphi(x) \cdot k$ . If  $X$  admits an equivariant cross section, let  $S = \varphi^{-1}(1)$ , and endow  $S$  with the trivial action. Then  $\Theta : X \rightarrow G \times S$ ,  $\Theta(x) = (\varphi(x), x \cdot \varphi(x)^{-1})$ , is a  $G$ -equivariant isomorphism, with inverse  $(k, s) \mapsto s \cdot k : G \times S \rightarrow X$ . As the action of  $G$  on  $G$  is linearly reductive, we deduce that the action of  $G$  on  $X$  is linearly reductive. In Theorem 2.11 we show that if the group  $G$  is unipotent, all the linearly reductive actions are of the above form.

The following particular case of the situation of Example 2.7, (2) is illustrative. In particular, it shows that if the action of a non linearly reductive affine group in a variety is linearly reductive, it cannot have fixed points.

**Corollary 2.8.** *Let  $G$  be an affine algebraic group and  $\varphi : R \rightarrow \mathbb{k}$  a morphism of rational  $G$ -module algebras and assume that  $\mathbb{k}$  is endowed with the trivial action. If the action of  $G$  on  $R$  is linearly reductive, then  $G$  is linearly reductive. In particular if the group  $G$  acts linearly reductively on an affine variety with a fixed point, then  $G$  is linearly reductive.*

*Proof.* We use Example 2.7, (2) to guarantee that the action of  $G$  on  $\mathbb{k}$  is linearly reductive. Hence  $G$  is linearly reductive.  $\square$

We can get some geometrical information about the closed orbits of linearly reductive actions.

**Corollary 2.9.** (1) *Let  $G$  be an affine algebraic group, and  $H \subseteq G$  a closed subgroup such that the homogeneous space  $G/H$  is affine. If  $G$  acts in a linearly reductive way on an affine variety  $X$  and  $H$  has a fixed point on  $X$ , then  $H$  is linearly reductive.*  
 (2) *Let  $G$  be an affine algebraic group acting in a linearly reductive way on an affine variety  $X$  and suppose that the action is separable. If  $Y \subseteq X$  is an affine orbit, then  $Y$  is  $G$ -equivariantly isomorphic to  $G/H$  where  $H$  is linearly reductive. In particular, if the base field has characteristic zero, any affine orbit of a linearly reductive action is of the form  $G/H$  for  $H$  reductive.*

*Proof.* (1) Using the a transitivity result that will be proved in Section 4 –see Theorem 4.1– we deduce that the action of  $H$  on  $X$  is also linearly reductive. As  $H$  has a fixed point on  $X$  we conclude from Corollary 2.8 that  $H$  is linearly reductive.  
 (2) The orbit  $Y$  is  $G$ -isomorphic to  $G/H$  that is an affine variety. As the action of  $G$  on  $X$  is linearly reductive so is the action of  $G$  on  $G/H$ . As the point  $1H \in G/H$  is  $H$ -fixed, we deduce that  $H$  is linearly reductive.  $\square$

In accordance with a result we prove later (Theorem 4.7), the consideration of reductive actions in the case of an unipotent group is relevant to the understanding of the reductivity of the actions.

**Lemma 2.10.** *Assume that  $U$  is an unipotent affine algebraic group and that  $R$  is a rational  $U$ -module algebra.*

- (1) *Then, the action of  $U$  on  $R$  is linearly reductive if and only if there exists a multiplicative total integral  $\sigma : \mathbb{k}[U] \rightarrow R$ .*
- (2) *Moreover, in the above situation if  $M \in {}_{(U,R)}\mathcal{M}$ ,  $p_R$  and  $p_M$  are the Reynolds operators associated to  $R$  and  $M$ , we have that  $p_M(rm) = p_R(r)p_M(m)$ .*
- (3) *In particular  $p_R : R \rightarrow {}^G R$  is an algebra homomorphism.*

*Proof.* (1) In accordance with Theorem 2.4, the linear reductivity of the action of  $U$  on  $R$  is equivalent to the existence of a total integral  $\sigma_0 : \mathbb{k}[U] \rightarrow R$ . In the case of a unipotent group, the existence of a total integral implies the existence of a *multiplicative* total integral as it is proved for example in [8, Theorem 11.8.1].  
 (2) Write  $\chi_M(m) = \sum m_0 \otimes m_1$  and  $\chi_R(m) = \sum r_0 \otimes r_1$ , then  $p_M(rm) = \sum \sigma(S(r_1 m_1)) r_0 m_0 = \sum \sigma(S(r_1)) \sigma(S(m_1)) r_0 m_0 = p_R(r) p_M(m)$ .  
 (3) This result is a particular case of (2).  $\square$

The theorem that follows is a characterization of linearly reductive unipotent actions and its proof is related to the results of [2], and [8, Theorem 11.8.2].

**Theorem 2.11.** *Assume that  $U$  is an unipotent affine algebraic group and that  $X$  is an affine  $U$ -variety. Then, the action of  $U$  on  $X$  is linearly reductive if and only if, there exist a closed subvariety  $L$  of  $X$  and an isomorphism of algebraic varieties,  $\Theta : X \rightarrow L \times U$  that is  $U$ -equivariant when we endow  $L \times U$  with the action given by the right translation on  $U$  and the trivial action on  $L$ .*

*Proof.* The algebra morphism  $\sigma : \mathbb{k}[U] \rightarrow \mathbb{k}[X]$  induces a morphism of varieties  $\Phi : X \rightarrow U$  with the property that  $\Phi(x \cdot u) = \Phi(x)u$  for all  $x \in X$ ,  $u \in U$ . If we call  $L = \Phi^{-1}(1)$ , the existence of a morphism  $\Theta$  was shown in Example 2.7, (6). Conversely, in the situation of the existence of  $\Theta$ , we obtain an isomorphism  $\mathbb{k}[X] \cong \mathbb{k}[L] \otimes \mathbb{k}[U]$  and this implies that the action of  $U$  on  $X$  is linearly reductive (see Example 2.7, (3)).  $\square$

**Observation 2.12.**

- (1) In the algebraic version, if  $U$  is a unipotent group, then a pair  $(U, R)$  is linearly reductive if and only if there exists a decomposition  $R = \mathbb{k}[U] \otimes L$  for some algebra  $L$  where  $U$  acts trivially.
- (2) Notice that in the above situation all the orbits of the action are isomorphic with  $U$  and that the quotient variety  $X/U$  exists and coincides with  $L$ .
- (3) Moreover, the above Theorem 2.11 means that except for trivial factors the action of the group  $U$  on itself by right translations – that we have seen it is linearly reductive in Example 2.7, (5)– is the only linearly reductive action of a unipotent group.

## 3. THE CASE OF A GROUP AND A CLOSED SUBGROUP

We consider the particular case of a group and a closed subgroup acting by translations.

The *restriction* functor  $\text{Res}_G^K : {}_G\mathcal{M} \rightarrow {}_K\mathcal{M}$  –  $\text{Res}_G^K(M) := M|_K$  – has a right adjoint: the *induction* functor  $\text{Ind}_K^G : {}_K\mathcal{M} \rightarrow {}_G\mathcal{M}$ .

Explicitly, if  $M \in {}_K\mathcal{M}$  we endow  $\mathbb{k}[G] \otimes M \in {}_K\mathcal{M}$  with the diagonal  $K$ -module structure, where the action on the  $G$ -polynomials is given by  $(x \cdot f)(g) = f(gx)$  for all  $x \in K$ ,  $f \in \mathbb{k}[G]$ . Then  $\text{Ind}_K^G(M) = {}^K(\mathbb{k}[G] \otimes M)$ , with the  $G$ -action defined as:  $g \star (\sum f_i \otimes m_i) = \sum f_i \cdot g^{-1} \otimes m_i$  for  $g \in G$ ,  $f_i \in \mathbb{k}[G]$  and  $m_i \in M$  where  $(f \cdot g^{-1})(g') = f(g^{-1}g')$  for  $g' \in G$  (see [2] or [8, Section 6.6]).

Moreover, if  $\alpha : M_1 \rightarrow M_1 \in {}_K\mathcal{M}$ , then  $\text{Ind}_K^G(\alpha)$  is the restriction of  $\text{id} \otimes \alpha$ .

We list some of the basic properties of the induction functor that will be later used.

**Observation 3.1.** In the above context we have:

- (1)  ${}^G(\text{Ind}_K^G(M)) = {}^K(\mathbb{k}[G]^G \otimes M) = {}^KM$ .
- (2) The counit of the above adjunction is  $\varepsilon_M : {}^K(\mathbb{k}[G] \otimes M) \rightarrow M$ ;  $\varepsilon_M(\sum f_i \otimes m_i) := \sum f_i(1)m_i$ . Observe that if  $\sum f_i \otimes m_i \in {}^K(\mathbb{k}[G] \otimes M)$  then:  $\sum f_i(1)y \cdot m_i = \sum f_i(y^{-1})m_i$ , for all  $y \in K$ .
- (3) The *tensor identity* guarantees that for  $M \in {}_K\mathcal{M}$  and  $N \in {}_G\mathcal{M}$ , there is a natural isomorphism between  $\text{Ind}_K^G(M \otimes N|_K) \cong \text{Ind}_K^G(M) \otimes N$ . In particular if  $M = \mathbb{k}$  and  $N$  is an arbitrary rational  $G$ -module we have that:  $\text{Ind}_K^G(N|_K) \cong {}^K\mathbb{k}[G] \otimes N$ . Then  ${}^KN \cong {}^G({}^K\mathbb{k}[G] \otimes N)$ .
- (4) If  $R$  is a  $K$ -module algebra, then  $\text{Ind}_K^G(R)$  is also  $G$ -module algebra.

**Observation 3.2.** If  $K \subseteq G$  is a closed inclusion of affine algebraic groups, then  $K$  is *observable* in  $G$  if  $\varepsilon_M$  is surjective for all  $M \in {}_K\mathcal{M}$ , moreover the following are equivalent:

- (1)  $K$  is observable in  $G$ ;
- (2)  $G/K$  is a quasi-affine variety;
- (3) If  $\chi : K \rightarrow \mathbb{k}^*$  is a multiplicative character, there is a polynomial  $f \in \mathbb{k}[G]$  such that for all  $x \in K$ ,  $x \cdot f = \chi(x)f$  (see [1, 9] or [8, Observation 10.2.4]);
- (4) If  $0 \neq M \in {}_K\mathcal{M}$ , then  $\text{Ind}_K^G(M) \neq 0$ .
- (5) For all  $0 \neq I \in \mathbb{k}[G]$  stable ideal, then  ${}^KI \neq 0$ .

The proof of most of the above assertions can be found in the monograph [9] and also in [8, Chapter 10]. We only mention the proof that (4) implies (3). Assuming (4) take an arbitrary rational character  $\chi$  of  $K$  and call  $\mathbb{k}_{\chi^{-1}}$  the one dimensional representation associated to  $\chi^{-1}$ . A non zero element of  $\text{Ind}_K^G(\mathbb{k}_{\chi^{-1}})$  is a non zero polynomial on  $G$ , that is a  $K$ -semi invariant with character to  $\chi$ .

Notice also that from (3) it follows immediately that if  $K \subseteq G$  is unipotent, then it is observable.

**Theorem 3.3.** Let  $G$  be an affine algebraic group and  $K \subseteq G$  a closed subgroup. Then, the following conditions are equivalent.

- (1) The right action by translation of  $K$  on  $G$  is linearly reductive.
- (2) The functor  $\text{Ind}_K^G : {}_K\mathcal{M} \rightarrow {}_G\mathcal{M}$  is exact or equivalently the homogeneous space  $G/K$  is an affine variety.
- (3) The subgroup  $K$  is observable in  $G$ , and if  $I \subseteq {}^K\mathbb{k}[G]$  is an ideal of  ${}^K\mathbb{k}[G]$  such that  $I\mathbb{k}[G] = \mathbb{k}[G]$ , then  $I = {}^K\mathbb{k}[G]$ .

*Proof.* The equivalence of conditions (1) and (2) is a part of Theorem 2.4. The equivalence of the two conditions appearing in (2) is due to E. Cline, B. Parshall and L. Scott ([2], see also [8, Theorem 11.4.5, Theorem 11.6.7] for a more elementary proof). We prove next that (3) is equivalent to the other three. First, observe that condition (2) implies in particular that  $K$  is observable in  $G$ . Assuming (1) and taking  $I$  as in (3), if we write  $1 = \sum_{i=1}^n f_i g_i$  with  $f_i \in I$  and  $g_i \in \mathbb{k}[G]$ , then the map  $\varphi : \bigoplus_i \mathbb{k}[G] \rightarrow \mathbb{k}[G]$ ,  $\varphi(g_1, \dots, g_n) = \sum_{i=1}^n f_i g_i$  is surjective. Then, by the linear reductivity of the action of  $K$  on  $G$ , there exist elements  $\{\bar{g}_i \in {}^K \mathbb{k}[G] : i = 1, \dots, n\}$  such that  $\sum_{i=1}^n f_i \bar{g}_i = 1$ ; therefore,  $I = {}^K \mathbb{k}[G]$ . In order to prove the converse observe that since  $K$  is observable in  $G$  then  $G/K$  is quasi-affine. Therefore, for some  $0 \neq f \in {}^K \mathbb{k}[G]$ , the subvariety  $(G/K)_f$  is non empty and affine. Call  $I \subseteq {}^K \mathbb{k}[G]$  the ideal generated by the family  $\{f \in {}^K \mathbb{k}[G] : (G/K)_f \text{ is affine}\}$ . If  $x \in G$ , then  $x \cdot (G/K)_f = (G/K)_{f \cdot x^{-1}}$ . Hence, the ideal  $I$  is stable by left translations by elements of  $G$ . Hence,  $I\mathbb{k}[G]$  has no zeroes and thus  $I\mathbb{k}[G] = \mathbb{k}[G]$ . Using the hypothesis (4), we prove the existence a finite number of elements  $f_1, \dots, f_n \in {}^K \mathbb{k}[G]$ , with the property that generate the unit ideal in  ${}^K \mathbb{k}[G]$  and whose principal open subsets  $(G/K)_{f_i}$  are affine. It is well known that in this situation  $G/K$  is an affine variety (see [8, Theorem, 1.4.49]).  $\square$

**Example 3.4.** The condition that  $K$  is observable in  $G$  cannot be omitted in (4), Theorem 3.3. Indeed, if  $G \neq \{e\}$  is a reductive group and  $K = B \subsetneq G$  is a Borel subgroup, then  ${}^B \mathbb{k}[G] = \mathbb{k}$  and  $G/B \neq \{p\}$  is projective.

We finish this section with a characterization of the linear reductivity of the action by translations on an affine homogeneous space.

**Theorem 3.5.** *Let  $K \subseteq G$  be a closed inclusion of affine algebraic groups. Then, the action of  $K$  on  $G$  is linearly reductive and the action of  $G$  by translations on  $G/K$  is linearly reductive if and only if  $K$  is linearly reductive.*

*Proof.* Let  $f : M \rightarrow N$  be a surjective morphism of rational  $K$ -modules. By the linear reductivity of  $(K, G)$ , it follows that  $\text{Ind}_K^G(M) \rightarrow \text{Ind}_K^G(N)$  is a surjective morphism of  $(G, {}^K \mathbb{k}[G])$ -modules. Hence, the linear reductivity of  $(G, G/K)$  guarantees that  ${}^G(\text{Ind}_K^G(M)) \rightarrow {}^G(\text{Ind}_K^G(N))$  is also surjective and then  $f|_{\mathcal{K}_M} : {}^K M \rightarrow {}^K N$  is surjective and  $K$  is linearly reductive.

Now conversely, let  $K \subseteq G$  be an inclusion of affine algebraic groups, with  $K$  linearly reductive and take  $f : M \rightarrow N$  a surjective morphism of  $(G, {}^K \mathbb{k}[G])$ -modules. Then, from the linear reductivity of  $K$ , one deduces that  $G/K$  is an affine variety and it follows from the Mackey's imprimitivity theorem ([3, Theorem 2.7]) that  $\text{Ind}_K^G : {}^K \mathcal{M} \rightarrow ({}_{G, {}^K \mathbb{k}[G]}) \mathcal{M}$  is an equivalence of categories. Then, there exists a couple of  $K$ -modules  $M_0, N_0$  and a surjective morphism of  $K$ -modules  $g : M_0 \rightarrow N_0$  such that  $f = \text{Ind}_K^G(g) : \text{Ind}_K^G(M_0) = M \rightarrow \text{Ind}_K^G(N_0) = N$ . By the transfer principle, it follows that  ${}^G M = {}^K M_0$ ,  ${}^G N = {}^K N_0$  and  $f|_{G_M} = g|_{K_{M_0}}$ , and the surjectivity of  $f|_{G_M} : {}^G M \rightarrow {}^G N$  follows from the linear reductivity of  $K$ .  $\square$

#### 4. TRANSITIVITY RESULTS

Next we present a relative version of Matsushima's criterion. (see [16] or [8, Theorem 11.7.1] for the classical version).

**Theorem 4.1** (Generalized Matsushima's criterion). (1) *Let  $K \subseteq G$  be a closed inclusion of affine algebraic groups and  $R$  a rational  $G$ -module algebra. Suppose that the action of  $K$  on  $G$  by translations is linearly reductive. If the action of  $G$  on  $R$  is linearly reductive, then the action of  $K$  on  $R$  is linearly reductive.*

(2) *Let  $K \triangleleft G$  be a closed normal inclusion of affine algebraic groups and  $R$  a rational  $G$ -module algebra. Then, the action of  $G$  on  $R$  is linearly reductive if and only if the action of  $K$  on  $R$  is linearly reductive and the action of  $G/K$  on  ${}^K R$  is linearly reductive.*

*Proof.* (1) Let  $\varphi : M_1 \rightarrow M_2$  be a surjective morphism in  $({}_{K, R}) \mathcal{M}$  and take an arbitrary element  $m_2 \in {}^K M_2$ . It follows from the reductivity hypothesis on  $(K, G)$  that  $\text{id} \otimes \varphi : {}^K(\mathbb{k}[G] \otimes M_1) \rightarrow$



${}^K(\mathbb{k}[G] \otimes M_2)$  is a surjective morphism of rational  $G$ -modules. Hence,  $\text{id} \otimes \text{id} \otimes \varphi : R \otimes {}^K(\mathbb{k}[G] \otimes M_1) \rightarrow R \otimes {}^K(\mathbb{k}[G] \otimes M_2)$  is also a surjective morphism in the category  ${}_{(G,R)}\mathcal{M}$ . Consider the element  $1 \otimes 1 \otimes m_2 \in R \otimes {}^K(\mathbb{k}[G] \otimes M_2)$ . As the action of an element  $x \in G$  on  $\sum f_i \otimes m_{2,i} \in {}^K(\mathbb{k}[G] \otimes M_2)$  is defined as  $x \cdot (\sum f_i \otimes m_{2,i}) = \sum f_i \cdot x^{-1} \otimes m_{2,i}$ , it is clear that the element  $1 \otimes 1 \otimes m_2$  is  $G$ -fixed. By definition of the linear reductivity of  $G$  on  $R$ , we deduce that there is an element  $\xi = \sum_{ij} r_i \otimes f_{ij} \otimes m_{1,ij} \in R \otimes {}^K(\mathbb{k}[G] \otimes M_1)$  fixed by the action of  $G$ , and satisfying that

$$\sum_{ij} r_i \otimes f_{ij} \otimes \varphi(m_{1,ij}) = 1 \otimes 1 \otimes m_2. \quad (1)$$

We evaluate the middle term of  $\xi$  at  $1 \in G$  and call the resulting element  $m'_1 = \sum_{ij} f_{ij}(1)r_i m_{1,ij}$ . Then evaluating equation (1) also at  $1 \in G$ , we obtain the equality:  $\varphi(m'_1) = m_2$ . All that remains to prove is that the element  $m'_1 \in {}^K M_1$ . As the element  $\xi = \sum_{ij} r_i \otimes f_{ij} \otimes m_{1,ij}$  is  $G$ -fixed, for  $y \in K$  we deduce that  $\sum_{ij} y \cdot r_i \otimes f_{ij} \cdot y^{-1} \otimes m_{1,ij} = \sum_{ij} r_i \otimes f_{ij} \otimes m_{1,ij}$ . After evaluation at 1 and multiplication we deduce that  $m'_1 = \sum_{ij} r_i f_{ij}(1) m_{1,ij} = \sum_{ij} (y \cdot r_i) f_{ij}(y^{-1}) m_{1,ij} = \sum_{ij} (y \cdot r_i) f_{ij}(1) y \cdot m_{1,ij} = y \cdot m'_1$ , where the third equality is a consequence of the considerations at the beginning of Section 3, see Observation 3.1.

- (2) Assume that the action of  $K$  on  $R$  is linearly reductive and the action of  $G/K$  on  ${}^K R$  is linearly reductive. Given  $\varphi : M_1 \rightarrow M_2$  a surjective morphism in the category  ${}_{(G,R)}\mathcal{M}$  and take  $m_2 \in {}^G M_2 \subseteq {}^K M_2$  and using the linear reductivity of  $(K, R)$ , we prove the existence a certain element  $m_1 \in {}^K M_1$  with the property that  $\varphi(m_1) = m_2$ . Call  $N_1 \subseteq {}^K M_1$  the  $(G/K, {}^K R)$ -module generated by  $m_1$  in  ${}^K M_1$  and  $N_2 \subseteq {}^K M_2$  the  $(G/K, {}^K R)$ -module generated by  $m_2$  in  ${}^K M_2$ . We can restrict  $\varphi$  to a surjective morphism of  $(G/K, {}^K R)$ -modules, that we continue calling  $\varphi : N_1 \rightarrow N_2$ . In this situation, using the linear reductivity of the action of  $G/K$  on  ${}^K R$ , applied to the map  $\varphi : N_1 \rightarrow N_2$  and to the element  $m_2 \in {}^{G/K} N_2$ , we find an element  $m_1 \in {}^{G/K}({}^K N_1) = {}^G N_1$ , such that  $\varphi(m_1) = m_2$ . Conversely, assume that  $(G, R)$  is linearly reductive, using Theorem 4.1 we deduce that  $(K, R)$  is linearly reductive. To prove the remainder part consider a surjective morphism of  $(G/K, {}^{G/K} R)$ -modules  $\varphi : M_1 \rightarrow M_2$  and  $m_2 \in {}^{G/K} M_2$ . Extend the action to view the map  $\varphi$  as living in the category of  $G$ -modules and consider the surjective morphism  $\text{id} \otimes \varphi : R \otimes M_1 \rightarrow R \otimes M_2 \in {}_{(G,R)}\mathcal{M}$ , and  $1 \otimes m_2 \in {}^G(R \otimes M_2)$ . Using the hypothesis we deduce the existence of an element  $\sum r_i \otimes m_{1,i} \in {}^G(R \otimes M_1)$  such that  $\sum r_i \otimes \varphi(m_{1,i}) = 1 \otimes m_2$ . Applying the Reynolds operator  $p_R : R \rightarrow {}^K R$ , we deduce that  $\sum \varphi(p_R(r_i) m_{1,i}) = \sum p_R(r_i) \varphi(m_{1,i}) = m_2$ . The element  $\sum p_R(r_i) m_{1,i} \in M_1$  is fixed by the action of  $G/K$  and hence, the proof is finished (see Observation 2.6,(3)).  $\square$

#### Observation 4.2.

- (1) The second assertion of Theorem 4.1 can also be proved using that for a rational  $G$ -module  $V$ , the *Inflation–Restriction sequence* for the rational cohomology groups:

$$0 \rightarrow H^1(G/K, {}^K(R \otimes V)) \xrightarrow{\text{Inf}} H^1(G, R \otimes V) \xrightarrow{\text{Res}} H^1(K, R \otimes V) \quad (2)$$

is exact: see [10, 11, 15].

Indeed, in the hypothesis of the above theorem using Theorem 2.4, we conclude that the first and the last terms of the sequence are zero. Hence, the middle term is also zero and that guarantees – by the same Theorem 2.4 – that the action of  $G$  on  $R$  is linearly reductive.

- (2) The converse of the first part of Theorem 4.1 is false, as the case  $G = \mathbb{k}^* \times \mathbb{k}$ ,  $K = \mathbb{k}^* \times \{0\}$  and  $R = \mathbb{k}$  shows.
- (3) In particular Theorem 4.1 above provides a proof of the following well known assertion: let  $K \subseteq G \subseteq H$  be a tower of inclusions of closed affine algebraic groups. If the homogeneous spaces  $G/K$  and  $H/G$  are affine, then  $H/K$  is also affine.
- (4) In geometric terms Theorem 4.1 can be formulated as follows (see also Theorem 6.3).

- (1) Let  $K \subseteq G$  be a closed inclusion of affine algebraic groups and  $X$  an affine  $G$ -variety. If the action of  $G$  on  $X$  is linearly reductive and the homogeneous space  $G/K$  is affine, then the action of  $K$  on  $X$  is linearly reductive.
- (2) Let  $G$  be an affine algebraic group and  $K \triangleleft G$  a closed normal subgroup. Let  $X$  be a affine  $G$ -variety such that (i) the action of  $K$  on  $X$  is linearly reductive; (ii) the quotient variety  $X/K$  exists and it is affine and the action of  $G/K$  on  $X/K$  is also linearly reductive. Then the action of  $G$  on  $X$  is linearly reductive. Conversely, if the action of  $G$  on  $X$  is linearly reductive and the quotient variety  $X/K$  exists and it is affine, then the action of  $G/K$  in  $X/K$  is linearly reductive.
- (5) In the case that  $X$  is a point, or alternatively that  $R$  is the base field, the above results read as follows.

*Let  $K \triangleleft G$  is a normal subgroup of the affine algebraic group  $G$ . Then  $K$  and  $G/K$  are linearly reductive if and only if  $G$  is linearly reductive.*

As an immediate application of Theorem 4.1 to the case when  $R = \mathbb{k}$ , we obtain the classical Matsushima's criterion in characteristic zero.

**Corollary 4.3** (Classical Matsushima's criterion). *Let  $G$  be a reductive group and  $K \subseteq G$  be closed subgroup and assume that  $\mathbb{k}$  has characteristic zero. Then  $G/K$  is an affine variety if and only if  $K$  is reductive.*

**Corollary 4.4.** *Consider a tower  $K \triangleleft G \subseteq L$  of closed subgroups of an affine algebraic group  $L$  such that the first inclusion is normal. Then, if the quotient variety  $L/K$  is affine and the action of  $G/K$  on  $L/K$  is linearly reductive, then the homogeneous space  $L/G$  is affine and conversely.*  $\square$

The following theorem becomes natural in the context of considerations of transitivity.

**Theorem 4.5.** *Let  $K \subseteq G$  be an inclusion of affine algebraic groups and assume that  $K$  is exact in  $G$ . Let  $R$  be a  $K$ -module algebra and consider the  $G$ -module algebra  $\text{Ind}_K^G(R)$ . If the action of  $G$  on  $\text{Ind}_K^G(R)$  is linearly reductive, so is the action of  $K$  on  $R$ .*

*Proof.* Assume that  $\varphi : M \rightarrow N$  is a surjective morphism in the category of  $(K, R)$ -modules. By the exactness hypothesis the morphism  $\text{Ind}(\varphi) : \text{Ind}_K^G(M) \rightarrow \text{Ind}_K^G(N)$  is also a surjective morphism in the category of  $(G, \text{Ind}_K^G(R))$ -modules. As for any rational  $K$ -module  $M$  the  $K$ -invariants of  $M$  and the  $G$ -invariants of  $\text{Ind}_K^G(M)$  are related by the equality  ${}^G(\text{Ind}_K^G(M)) = {}^K M$  the result follows directly.  $\square$

Let  $K \subseteq G$  be a closed inclusion of affine algebraic groups and assume that  $X$  is an affine  $K$ -variety. In this context we can form the *induced variety*  $X *_K G$ , that is the geometric quotient of  $X \times G$  by the action  $K \times (X \times G) \rightarrow X \times G$ ,  $a \cdot (x, g) = (x \cdot a, a^{-1}g)$ . It is well known that this quotient exists, and has a natural structure of  $G$ -variety. Applying Theorem 4.5, we obtain some insight in the relationship between the reductivity of the action of  $K$  on  $X$  and the action of  $G$  in  $G *_K X$ .

**Theorem 4.6.** *Let  $K \subseteq G$  be an inclusion of affine algebraic groups and assume that  $K$  is exact in  $G$ . Let  $X$  be an affine  $K$ -variety. Then if  $X *_K G$  is affine, and the action of  $G$  on  $X *_K G$  is linearly reductive, then the action of  $K$  on  $X$  is also linearly reductive.*

*Proof.* Since  $K$  is exact in  $G$ ,  $G/K$  is affine, and using [20, Lemma 3.16] it follows that  $X *_K G$  is affine. The rest of the assertion is just the geometric version of Theorem 4.5.  $\square$

From the above transitivity results (Theorem 4.1,(1)) we obtain the following characterization of linearly reductive actions.

**Theorem 4.7.** *Let  $G$  be an affine algebraic group and  $\mathcal{R}_u(G)$  its unipotent radical. Assume that  $X$  is an affine variety equipped with a regular action of  $G$ . Then the action of  $G$  on  $X$  is linearly reductive if and only if the action of  $\mathcal{R}_u(G)$  on  $X$  is linearly reductive. Similarly for the situation that  $R$  is a rational  $G$ -module algebra.*

*Proof.* If the action of  $G$  on  $X$  is linearly reductive, it follows from Theorem 4.1, (1) that the action of  $\mathcal{R}_u(G)$  is also linearly reductive. Conversely, in the case that  $\mathcal{R}_u(G)$  acts in a linearly reductive way on  $R$ , as the quotient group  $G/\mathcal{R}_u(G)$  is reductive our conclusion follows from Theorem 4.1, (2).  $\square$

## 5. LEVI DECOMPOSITION OF A LINEARLY REDUCTIVE ACTION

According the Theorem 4.7, the linear reductivity of the action of the affine algebraic group  $G$  on the affine variety  $X$  is determined by the linear reductivity of the action of the unipotent radical  $\mathcal{R}_u(G)$ . In this case, we can complement the results of Theorem 2.11, and obtain a  $\mathcal{R}_u(G)$ -decomposition of the variety  $X$  that is similar to the classical decomposition of an affine algebraic group as a semi-direct product of the unipotent radical and a closed reductive subgroup – the Levi decomposition.

**Definition 5.1.** (1) Let  $G$  be an affine algebraic group and  $X$  be an affine  $G$ -variety. A *Levi decomposition* for the  $G$ -variety  $X$  is an  $\mathcal{R}_u(G)$ -isomorphism  $X \cong \mathcal{R}_u(G) \times L_G(X)$  where  $L_G(X)$  is an affine variety endowed with the trivial  $\mathcal{R}_u(G)$ -action. In this situation the variety  $L_G(X)$  is called a Levi factor and it is clear that the projection  $X \rightarrow L_G(X)$  is the geometric quotient of  $X$  by the  $\mathcal{R}_u(G)$ -action.

(2) Let  $G$  be an affine algebraic group and  $R$  a  $G$ -module algebra. A Levi decomposition of  $R$  is an isomorphism of  $\mathcal{R}_u(G)$ -module algebras  $R \cong \mathbb{k}[\mathcal{R}_u(G)] \otimes L$ , where  $L$  is an algebra with trivial  $\mathcal{R}_u(G)$ -action.

**Theorem 5.2.** (1) *Let  $G$  be an affine algebraic group and  $X$  is an affine  $G$ -variety. Then the action of  $G$  on  $X$  is linearly reductive if and only if  $X$  admits a Levi decomposition.*

(2) *Let  $G$  be an affine algebraic group and  $R$  a  $G$ -module algebra. Then  $(G, R)$  is a linearly reductive pair if and only if  $R$  admits a Levi decomposition.*

*Proof.* (1) Assume that the action of  $G$  on  $X$  is linearly reductive, then as  $\mathcal{R}_u(G)$  is a normal subgroup, it follows from previous transitivity results – see Theorem 4.7 – that the action of  $\mathcal{R}_u(G)$  on  $X$  is also linearly reductive. Then, the existence of a Levi decomposition follows directly from Theorem 2.11.

Conversely, in the case there is a decomposition of  $X$  as in Definition 5.1, from the fact that the action of  $\mathcal{R}_u(G)$  is trivial on the Levi factor, it follows that the action of this radical is linearly reductive on  $X$  (see Theorem 2.11). It follows by transitivity (see Theorem 4.7) that the action of  $G$  on  $X$  is also linearly reductive.

(2) We omit this part of the proof as it is standard –see Observation 2.12–.  $\square$

**Observation 5.3.** In the case that we consider an inclusion of the form:  $\mathcal{R}_u(G) \triangleleft G$  and the action of the radical by translations of  $G$ , the multiplicative integral  $\sigma : \mathbb{k}[\mathcal{R}_u(G)] \rightarrow \mathbb{k}[G]$  and the associated morphism  $\Phi : G \rightarrow \mathcal{R}_u(G)$  satisfy the following properties:

- (1) For all  $x \in G$ ,  $u \in \mathcal{R}_u(G)$ ,  $f \in \mathbb{k}[\mathcal{R}_u(G)]$ :  $\sigma(u \cdot f) = u \cdot \sigma(f)$ ,  $\Phi(xu) = \Phi(x)u$ .
- (2) For all  $x, y \in G$ ,  $f \in \mathbb{k}[\mathcal{R}_u(G)]$ :  $\sigma(x \cdot f \cdot x^{-1}) = x \cdot \sigma(f) \cdot x^{-1}$ ,  $\Phi(xyx^{-1}) = x\Phi(y)x^{-1}$ .
- (3) For all  $x \in G$ ,  $u \in \mathcal{R}_u(G)$ ,  $f \in \mathbb{k}[\mathcal{R}_u(G)]$ :  $\sigma(f \cdot u) = \sigma(f) \cdot u$ ,  $\Phi(ux) = u\Phi(x)$ .

The first property is just the  $\mathcal{R}_u(G)$ -equivariance of the maps  $\sigma$  and  $\Phi$ ; the second follows in a standard manner from the uniqueness of the integral and the third is an easy consequence of the second.

Call  $L = \{\ell \in G : \Phi(\ell) = 1\}$  and endow it with the following product:  $\ell \star t = \Phi(\ell t)^{-1} \ell t$ . It follows easily from the properties (1),(2),(3) that the product thus defined is associative.

Define the product on the set  $\mathcal{R}_u(G) \times L$ :

$$(u, \ell) \cdot_{\Phi} (v, s) := (u(\ell v \ell^{-1}) \Phi(\ell s), \ell \star s). \quad (3)$$

A direct computation shows that the product defined above is associative and that the map:

$$(u, \ell) \mapsto u\ell : (\mathcal{R}_u(G) \times L, \cdot_{\Phi}) \rightarrow G,$$

is a group isomorphism.

Suppose that the map  $\Phi : G \rightarrow \mathcal{R}_u(G)$  satisfies the following cocycle condition:

$$\Phi(xy) = \Phi(y)y^{-1}\Phi(x)y = \Phi(y)y.\Phi(x).$$

In this situation it is clear that in  $L$  the product  $\star$  coincides with the product induced by  $G$  –as  $\Phi(x) = \Phi(y) = 1$  implies that  $\Phi(xy) = 1$ –, and also that the product  $\cdot_\Phi$  coincides with the usual semidirect product on  $\mathcal{R}_u(G) \times L$ . It is also clear that  $\bar{\Phi} : G/\mathcal{R}_u(G) \rightarrow L$  is an isomorphism.

The proof that in the case of characteristic zero, the map  $\Phi$  can be taken as to satisfy the cocycle condition, is the main content of the classical Levi decomposition –see [8] for a proof or [13, 19] for the cohomological viewpoint and counterexamples (in positive characteristic)–.

**Observation 5.4.** We recall some basic facts on extensions of actions. Let  $G$  be an abstract group and  $H \triangleleft G$  a normal subgroup.

Consider an arbitrary set  $L$ , and the right action of  $H$  on  $H \times L$  given by the right translation on  $H$  and the trivial action on  $L$ . An extension of this action to an action of  $G$  on  $H \times L$  (that we write as  $((r, \ell), g) \mapsto (r, \ell) \cdot g : (H \times L) \times G \rightarrow H \times L$ ), is the same than: a) an action  $\star : L \times G \rightarrow L$  that is trivial on  $L$  ( $\ell \star r = \ell, r \in H, \ell \in L$ ); b) a cocycle  $c : L \times G \rightarrow H$  satisfying the following conditions:

$$c(\ell, gg') = c(\ell \star g, g')g'^{-1}c(\ell, g)g' \quad ; \quad c(\ell, h) = h. \quad (4)$$

for all  $\ell \in L, h \in H, g, g' \in G$ .

Indeed, given  $\star$  and  $c$  as above, the action can be defined as  $(r, \ell) \cdot g := (c(\ell, hg), \ell \star g)$  for  $\ell \in L, h \in H, g \in G$ . Conversely, if we write  $(h, \ell) \cdot g = (1, \ell) \cdot (hg) = (1, \ell) \cdot g \cdot (g^{-1}hg)$ , it is clear that if we define  $(c(\ell, g), \ell \star g) := (1, \ell) \cdot g$ , we obtain our conclusion.

Moreover, if we consider equation (4) for the case that  $g = h \in H$ , then for all  $h \in H, g \in G$  we have:

$$c(\ell, hg) = c(\ell, g)g^{-1}hg. \quad (5)$$

This equality implies that for  $\ell \in L, h \in H, g, g' \in G$ :

$$c(\ell \star g, g')g'^{-1}c(\ell, g)g' = c(\ell \star g, c(\ell, g)g'). \quad (6)$$

Then, the conditions given in equation (4) are also equivalent to:

$$c(\ell, gg') = c(\ell \star g, c(\ell, g)g') \quad ; \quad c(\ell, h) = h \quad (7)$$

for all  $\ell \in L, h \in H, g, g' \in G$ .

From the above considerations we deduce the following explicit description of the linearly reductive actions of an affine algebraic group  $G$  on the affine variety  $X$ .

**Theorem 5.5.** *Let  $G$  be an affine algebraic group and  $X$  is an affine  $G$ -variety. Then the action of  $G$  on  $X$  is linearly reductive if and only if  $X$  is  $G$ -equivariantly isomorphic with a variety of the form  $\mathcal{R}_u(G) \times L$  where  $L$  is an affine variety. The action of  $G$  on  $\mathcal{R}_u(G) \times L$  is given in terms of a pair  $(\star, c)$ , where  $\star : L \times G \rightarrow L$  is a regular right action, trivial when restricted to  $\mathcal{R}_u(G)$ , and  $c : L \times G \rightarrow \mathcal{R}_u(G)$  is a morphism of varieties satisfying the following cocycle condition:*

$$c(\ell, gg') = c(\ell \star g, g')g'^{-1}c(\ell, g)g' = c(\ell \star g, c(\ell, g)g') \quad ; \quad c(\ell, r) = r \quad (8)$$

where  $\ell \in L, g, g' \in G$  and  $r \in \mathcal{R}_u(G)$ .

In this situation the action of  $G$  on  $\mathcal{R}_u(G) \times L$  is given by the formula

$$(r, \ell) \cdot g = (c(\ell, rg), \ell \star g) = (c(\ell, g)g^{-1}rg, \ell \star g), \quad (9)$$

where  $r \in \mathcal{R}_u(G), \ell \in L$  and  $g \in G$

*Proof.* The proof follows directly from Theorem 5.2 and Observation 5.4. □

Hence, in the above perspective, a linearly reductive action of a group  $G$  on an affine variety  $X$ , is built from two pieces, as in equation (9). One is a classical GIT action  $\star$  of a *reductive* group  $G/\mathcal{R}_u(G)$  on an affine variety  $L$  and the other is a rather more intricate map defined in terms of a polynomial cocycle  $c : L \times G \rightarrow \mathcal{R}_u(G)$  compatible with  $\star$  in the sense of equation (8).

## 6. INVARIANTS AND QUOTIENTS

Even though a group that has a linearly reductive action on an affine variety  $X$  need not have a trivial unipotent radical, as we have obtained a good control on the linearly reductive actions of the unipotent groups (Theorem 2.11) most of the “classical” results on invariants and quotients of reductive groups can be recuperated in the relative context.

**Theorem 6.1.** *Assume that  $R$  is a  $\mathbb{k}$ -finitely generated rational commutative  $G$ -module algebra and that the action of  $G$  on  $R$  is linearly reductive. Let  $S$  be an algebra object in  ${}_{(G,R)}\mathcal{M}$  that is finitely generated over  $S$ , then  ${}^G S$  is finitely generated over  $\mathbb{k}$ .*

*Proof.* We may assume that  $S = R$  because in the above context, the action of  $G$  on  $S$  is linearly reductive –see Example 2.7 (2)– and clearly  $S$  is a finitely generated  $\mathbb{k}$ -algebra. It follows from the generalized Matsushima criterion (c.f. Theorem 4.1), that the action of  $U = \mathcal{R}_u(G)$  on  $R$  is linearly reductive. Then, using the multiplicative surjective morphism  $p_R : R \rightarrow {}^U R$  –see Lemma 2.10, (3)– we conclude that the algebra of invariants  ${}^U R$  is finitely generated over  $\mathbb{k}$ . Now, as  ${}^G R = {}^{G/U}({}^U R)$ , using the classical results about invariants of reductive groups, we conclude what we want.  $\square$

**Theorem 6.2.** *Assume that  $R$  is a rational commutative  $G$ -module algebra, that the action of  $G$  on  $R$  is linearly reductive and let  $S$  be an algebra object in  ${}_{(G,R)}\mathcal{M}$ .*

- (1) *If  $I \subseteq {}^G S$  is an ideal, then  $I = IS \cap {}^G S$ ;*
- (2) *If  $J_1, J_2 \subset S$  are  $G$ -stable ideals, then:*

$$J_1 \cap {}^G S + J_2 \cap {}^G S = (J_1 + J_2) \cap {}^G S;$$

- (3) *If  $J_1 + J_2 = S$ , then  $J_1 \cap {}^G S + J_2 \cap {}^G S = {}^G S$ .*

*Proof.* (1) Clearly  $I \subseteq IS \cap {}^G S$ . If we take  $s = \sum_t i_t s_t \in IS \cap {}^G S$  with  $i_t \in I, s_t \in S$  and apply the Reynolds operator  $p_S$  we obtain that  $s = p_S(s) = \sum i_t p_S(s_t) \in I {}^G S = I$ .  
 (2) It is clear that  $J_1 \cap {}^G S + J_2 \cap {}^G S \subseteq (J_1 + J_2) \cap {}^G S$ . If we take an element  $s_1 + s_2 \in (J_1 + J_2) \cap {}^G S$ , and consider  $s_1 + J_1 \cap J_2 \in J_1/(J_1 \cap J_2)$  and  $s_2 + J_2 \cap J_2 \in J_2/(J_1 \cap J_2)$ , from the equality  $x \cdot (s_1 + s_2) = (s_1 + s_2)$  we deduce that  $x \cdot s_1 - s_1 = s_2 - x \cdot s_2 \in J_1 \cap J_2$ . In other words, we have that  $s_1 + J_1 \cap J_2 \in {}^G(J_1/(J_1 \cap J_2))$  and  $s_2 + J_1 \cap J_2 \in {}^G(J_2/(J_1 \cap J_2))$ . Using the linear reductivity of the action we find  $\ell_1 \in {}^G J_1$  and  $\ell_2 \in {}^G J_2$  such that  $\ell_1 - s_1 \in J_1 \cap J_2$  and  $\ell_2 - s_2 \in J_1 \cap J_2$ . Then we write  $s_1 + s_2 = \ell_1 + \ell_2 + t$  with  $t \in {}^G(J_1 \cap J_2)$ . Hence,  $s_1 + s_2 \in {}^G J_1 + {}^G J_2$ .  
 (3) This condition follows immediately from the previous one.  $\square$

The semi-geometric quotient (in particular the categorical quotient) of a linearly reductive action exists.

Once that Theorems 6.1 and 6.2 are established, the proof goes along the same lines than the standard proof for actions of reductive groups on affine varieties and for that reason we omit it (see for example [8][Theorem 13.2.4] or [18][Theorem 3.4]).

**Theorem 6.3.** *Assume that the regular action of an affine algebraic group  $G$  on the affine variety  $X$  is linearly reductive. Consider the affine variety  $Y$  having as algebra of polynomial functions  ${}^G \mathbb{k}[X]$  and call  $\pi : X \rightarrow Y$  the associated morphism. Then, the pair  $(Y, \pi)$  is a semi-geometric quotient of  $X$  by  $G$ .  $\square$*

Hence, we have shown that if we have a linearly reductive action of a group  $G$  on an affine variety  $X$ , to construct the quotient variety we can first take the quotient  $X/U$ , being  $U$  the unipotent radical of  $G$  – this quotient will be affine because of Theorem 2.11 –; and then use the classical results for quotients of affine varieties by reductive groups in order to obtain  $X/G = \frac{X/U}{G/U}$ .

## 7. UNIPOTENT ACTIONS

Assume that  $G$  is an affine algebraic group and  $R$  a left rational  $G$ -module algebra. In this section we define the concept of *unipotent action* in terms of the fixed point functor:  $M \mapsto {}^G M : (G, R)\mathcal{M} \rightarrow {}_{G,R}\mathcal{M}$ . Along this section we use some of the results and definitions recalled at the beginning of Section 3, particularly the ones in Observation 3.2.

**Definition 7.1.** Let  $G$  and  $R$  be as above. We say that *the action of  $G$  on  $R$  is unipotent*, or that *the pair  $(G, R)$  is unipotent*, if for all  $0 \neq M \in (G, R)\mathcal{M}$ , then  ${}^G M \neq 0$ .

In the case that  $R = \mathbb{k}[X]$  for some affine variety  $X$  acted regularly by  $G$  on the right, we say that *the action of  $G$  on  $X$  is unipotent*, or that *the pair  $(G, X)$  is unipotent*, if and only if the action of  $G$  on  $\mathbb{k}[X]$  is unipotent.

It is clear that in the case that  $X = \{\star\}$  or that  $R = \mathbb{k}$ , if the action of  $G$  on any of these is unipotent, then  $G$  is a unipotent group.

Also, the action of  $G$  on itself by left translations is unipotent. Indeed, the objects  $M \in (G, \mathbb{k}[G])\mathcal{M}$  have the form  $M = \mathbb{k}[G] \otimes {}^G M$  –see Example 2.7–, and this implies that  $M \neq 0$  if and only if  ${}^G M \neq 0$  and hence the pair  $(G, G)$  is unipotent.

**Lemma 7.2.** Let  $G$  be an affine algebraic group and  $\varphi : R \rightarrow S$  a homomorphism of rational  $G$ -module algebras. Suppose the action of  $G$  on  $R$  is unipotent, then the action of  $G$  on  $S$  is unipotent. Similarly, if  $f : X \rightarrow Y$  is an equivariant  $G$ -morphism of affine varieties and the action of  $G$  on  $Y$  is unipotent, so is the action of  $G$  on  $X$ .

*Proof.* Assume that  $M \in (G, S)\mathcal{M}$  is a non zero object. We may change scalars and view it as an object in  $(G, R)\mathcal{M}$  that – by hypothesis – has non zero fixed part. Then, by definition the action of  $G$  on  $S$  is unipotent.  $\square$

We have the following transitivity result that yields in the particular case of a group and a subgroup, the equivalence of the concept of observability with that of unipotent action.

**Theorem 7.3.** Assume that  $K \subseteq G$  is a closed inclusion of affine algebraic groups.

- (1) Let  $R$  be a rational  $G$ -module algebra and assume that  $K$  is observable in  $G$ . If the action of  $G$  on  $R$  is unipotent, the same is true for the action of  $K$  on  $R$ . Similarly if the action of  $G$  on an affine variety  $X$  is unipotent, so is the action of  $K$ .
- (2) The action of  $K$  on  $G$  is unipotent if and only if  $K$  is observable in  $G$ .

*Proof.* (1) If  $M$  is a non-zero object in  $(K, R)\mathcal{M}$ , then  $\text{Ind}_K^G(M)$  an object in the category of  $(G, \text{Ind}_K^G(R))$ -modules that in accordance with Observation 3.2 is not zero. Using the tensor identity and the fact that  $R$  is a  $G$ -module –see observation 3.1– we deduce that  $\text{Ind}_K^G(R) = {}^K \mathbb{k}[G] \otimes R$ . Hence, one can view  $\text{Ind}_K^G(M)$  as an  $R$ -module via the natural inclusion of  $R$  into  ${}^K \mathbb{k}[G] \otimes R$ . In explicit terms if  $\sum f_i \otimes m_i \in \text{Ind}_K^G(M) = {}^K (\mathbb{k}[G] \otimes M)$  and  $r \in R$ , then  $r \cdot (\sum f_i \otimes m_i) = \sum S(r_1) f_i \otimes r_0 \cdot m_i$ . Considering  $\text{Ind}_K^G(M)$  as a object in  $(G, R)\mathcal{M}$  and using the unipotency of  $G$  in  $R$  we conclude that  ${}^K M = {}^G \text{Ind}_K^G(M) \neq 0$ .

- (2) An inclusion  $K \subseteq G$  is observable if and only if for all  $K$ -stable ideals  $I \subseteq \mathbb{k}[G]$ ,  $I^K = 0$  implies that  $I = 0$  (see [8, Theorem 2.9]). Hence the condition of unipotency of the action implies the observability. Conversely, if  $K \subseteq G$  is observable, using the fact that  $G$  is unipotent in  $G$  and the transitivity result just proved, we conclude that  $K$  acts unipotently on  $G$ .  $\square$

**Observation 7.4.**

- (1) It is clear that in the case that  $R$  is the base field  $\mathbb{k}$ , the above concept of unipotent action coincides with the concept of unipotent group. In particular, it follows directly (or from Lemma 7.2) that if  $G$  is a unipotent group, then the action of  $G$  on an arbitrary algebra  $R$  is also unipotent.

- (2) Suppose that  $G$  is an affine algebraic group and  $R$  a  $G$ -module algebra such that  $(G, R)$  is unipotent. Assume there is a  $G$ -equivariant augmentation  $\varepsilon : R \rightarrow \mathbb{k}$ . Using Lemma 7.2 and the previous consideration, we deduce that the group  $G$  is unipotent. In particular if the group  $G$  acts unipotently on an affine variety and has a fixed point, then  $G$  is a unipotent group.
- (3) It follows from the fact that exact subgroups are observable and from Theorem 7.3, that in the case of an inclusion  $K \subseteq G$  with  $K$  exact in  $G$ , if a pair  $(G, R)$  is unipotent, the same happens with the pair  $(K, R)$ . In particular, this is true for the situation that  $K \triangleleft G$ .

Next we present a relative version of Kostant–Rosenlicht theorem concerning closed orbits of unipotent groups (see for example [21, 8, 12, 14] for the original result).

**Theorem 7.5** (Generalized Kostant–Rosenlicht theorem). *Let  $G$  be an affine algebraic group and  $X$  an affine  $G$ -variety. Assume that the action of  $G$  on  $X$  is unipotent, then all the orbits of  $G$  on  $X$  are closed. Moreover, if the action is separable, all the orbits are of the form  $G/H$  where  $H$  is an unipotent closed subgroup of  $G$ .*

*Proof.* Assume that  $O$  is an orbit of the action of  $G$  on  $X$ , call  $Y = \overline{O}$  and  $C = \overline{O} \setminus O$ . If  $O \neq \overline{O}$ , then  $C$  is closed non empty and  $G$ -stable,  $\emptyset \neq C \subseteq Y \subseteq X$ . By our assumption, we can find  $f \in \mathbb{k}[Y]$  that is zero on  $C$  and not zero in a point  $p \in O$ . Using Lemma 7.2 we may assume that  $Y = X$ . Consider the non zero  $G$ -stable ideal  $I$  of  $\mathbb{k}[X]$  generated by  $f$ . A generic element  $g \in I$  is of the form:  $g = \sum g_i(x_i \cdot f)$ , with  $g_i \in \mathbb{k}[X]$  and  $x_i \in G$ . Using the hypothesis of unipotency of the action of  $G$  on  $X$  we can guarantee the existence of element  $0 \neq g \in {}^G I$ . Then,  $g \in \mathbb{k}[X]$  is constant in the orbit  $O$  of  $p$ , and then it is constant in  $X$ , the closure of the orbit. It is clear by the construction of  $f$  that for an arbitrary point  $q \in C$ ,  $g(q) = \sum g_i(q)f(q \cdot x_i) = 0$ . Then,  $g = 0$  and this is a contradiction.

As to the proof of the last assertion we proceed as follows. Let  $Y$  be a (closed) orbit. Hence for some  $H \subseteq G$ , the homogeneous space  $G/H$  is an affine variety isomorphic to  $Y$ . It follows from Lemma 7.2 and Theorem 7.3, that the action of  $H$  on  $G/H$  is unipotent. Since  $H1$  is a fixed point on  $G/H$  for this action, we deduce that  $H$  is a unipotent group (see Observation 7.4 , (2)).  $\square$

**Observation 7.6.**

- (1) If  $G$  acts in a unipotent way on  $X = \{\star\}$ , then by Observation 7.4, (1) *all* the actions of  $G$  on arbitrary varieties are unipotent. Then, all the orbits of  $G$  are closed in accordance to the above generalized version of Kostant–Rosenlicht theorem and we recover the classical theorem.
- (2) The last assertion of Theorem 7.5 should be interpreted as a generalization of the fact that a closed subgroup of a unipotent group is always unipotent.

Recall the following definition that is useful in relation to the concept of observability (for the particular group–subgroup situation see the beginning of Section 3).

**Definition 7.7.** Let  $G$  be an affine algebraic group acting regularly on an affine variety  $X$ . We say that a character  $\chi : G \rightarrow \mathbb{k}^*$  is extendible (to  $X$ ), if there is a non zero  $\chi$ -semi-invariant polynomial in  $\mathbb{k}[X]$ . The multiplicative monoid of extendible characters will be denoted as  $\mathcal{E}_G(X)$  and the group of all characters as  $\mathcal{X}(G)$ .

Next lemma generalizes the fact that unipotent groups have no characters.

**Lemma 7.8.** *Let  $G$  be an affine algebraic group acting regularly on an affine variety  $X$ . If the action of  $G$  on  $X$  is unipotent, then  $\mathcal{E}_G(X) = \mathcal{X}(G)$ .*

*Proof.* Let  $\chi : G \rightarrow \mathbb{k}^* \in \mathcal{X}(G)$ , be a rational character. We call  $\mathbb{k}_\chi[X]$  the  $(G, \mathbb{k}[X])$ -module consisting of  $\mathbb{k}[X]$  endowed with the following “twisted”  $G$ -module structure: if  $f \in \mathbb{k}_\chi[X]$  and  $x \in G$ ,  $x \rightarrow_\chi f = \chi^{-1}(x)x \cdot f$  where  $(x, f) \mapsto x \cdot f$  is the original action of  $G$  on  $\mathbb{k}[X]$ . An element  $f \in {}^G \mathbb{k}_\chi[X]$  is a  $\chi$ -semi-invariant polynomial on  $X$ . Hence, all characters are extendible.  $\square$

Next we compare the concept of unipotent action with the concept of observable action as defined in [20].

**Definition 7.9.** ([20]) An action of an affine algebraic group  $G$  on an affine variety  $X$  is said to be *observable* if for every non zero  $G$ -stable ideal  $I \subseteq \mathbb{k}[X]$ , then  ${}^G I \neq \{0\}$ .

**Observation 7.10.**

- (1) It follows immediately Definition 7.1 that unipotent actions are observable. It is also clear that in the case of a group and a subgroup the action is observable if and only if the group  $K$  is observable in  $G$ . It is also clear that in this context, the concept of observable action and of unipotent action coincide (see 7.3).
- (2) In accordance with [20, Corollary 3.14] an action of a connected affine algebraic group  $G$  on a factorial variety  $X$  is observable if and only if the following two conditions hold: (i) there is an open subset of closed orbits of maximal dimension; (ii)  $\mathcal{E}_G(X)$  is a group.

In the presence of the stronger condition of unipotency of the action, the above properties are satisfied: (i) all the orbits are closed; (ii)  $\mathcal{E}_G(X)$  is the group of all characters.

- (3) The action of  $\mathbb{k}^*$  on  $\mathbb{k}^2$  given by  $(a, b) \cdot t = (at, bt^{-1})$  is not unipotent as it has non closed orbits but it is observable as follows from [20, Theorem 4.4] –also a direct verification shows the validity of conditions (i) and (ii) in this context.

Next we show that the only actions that are at the same time unipotent and linearly reductive, are trivial in the sense of Theorem 7.11.

**Theorem 7.11.** *Let  $G$  be an affine algebraic group acting in a separable way on the affine variety  $X$ . If the action of  $G$  on  $X$  is unipotent and linearly reductive, then all the orbits are closed and isomorphic to  $G$ .*

*Proof.* Consider an orbit  $O$  of  $G$  on  $X$ . The generalized theorem of Kostant–Rosenlicht –Theorem 7.5–guarantees that all orbits are closed and using Example 2.7, (2) and Lemma 7.2 we deduce that the action of  $G$  on  $O$  is also linearly reductive and unipotent. By the separability hypothesis, we deduce that the orbits are of the form  $G/G_x$  for some closed subgroup of  $G$  and using Theorem 3.5 and Theorem 7.5 we deduce that  $G_x$  is at the same time linearly reductive and unipotent. Hence it is trivial.  $\square$

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